A Few Lessons from QCD perturbative Analysis at Low Energies

[Divergent Series, Summation, Practical Alternatives]

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Power Series with Factorial Coefficients

Divergent series

$\sum_n n!\,(\alpha_s)^n$

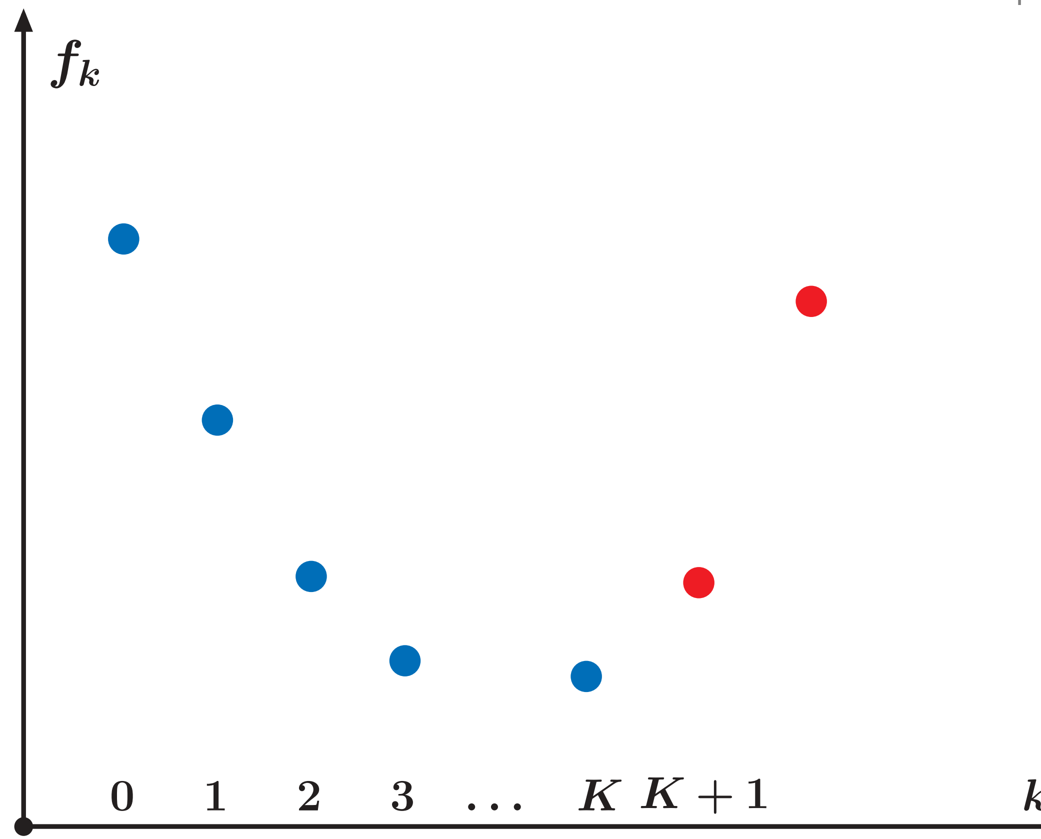
Finite Sum

$F_K(\alpha_s) = \sum_n^K f_n;$

$f_n = n!\,(\alpha_s)^n$

Poincaré estimate

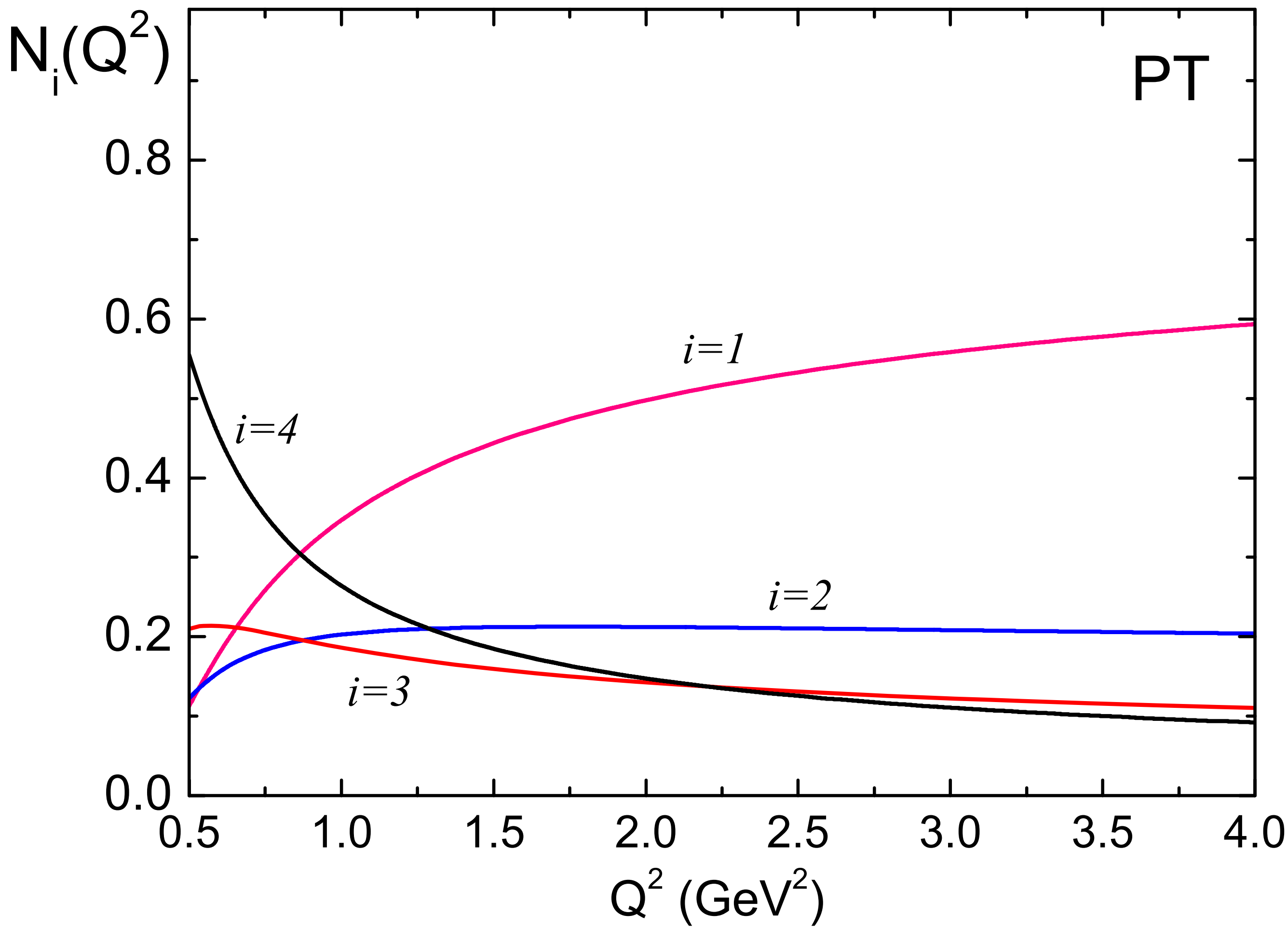
$\Delta F(\alpha_s) \sim f_K$



Optimal number of terms $K_* \sim 1/\alpha_s$ for numerical estimation with lower limit of possible accuracy, $\mathbf{f_{K_*}}$

4-loop Evidence from Bjorken Sum Rule

of the PT series "blow up" at $Q^2 \lesssim 2 - 3\,\mathrm{GeV}^2$



Relative weight of 1-, 2-, 3-, 4-loop terms.

Asymptotic series born by Essential Singularity $e^{-1/g}$

This singularity is usual in Physics of Big Systems (representable via Functional Integral) :

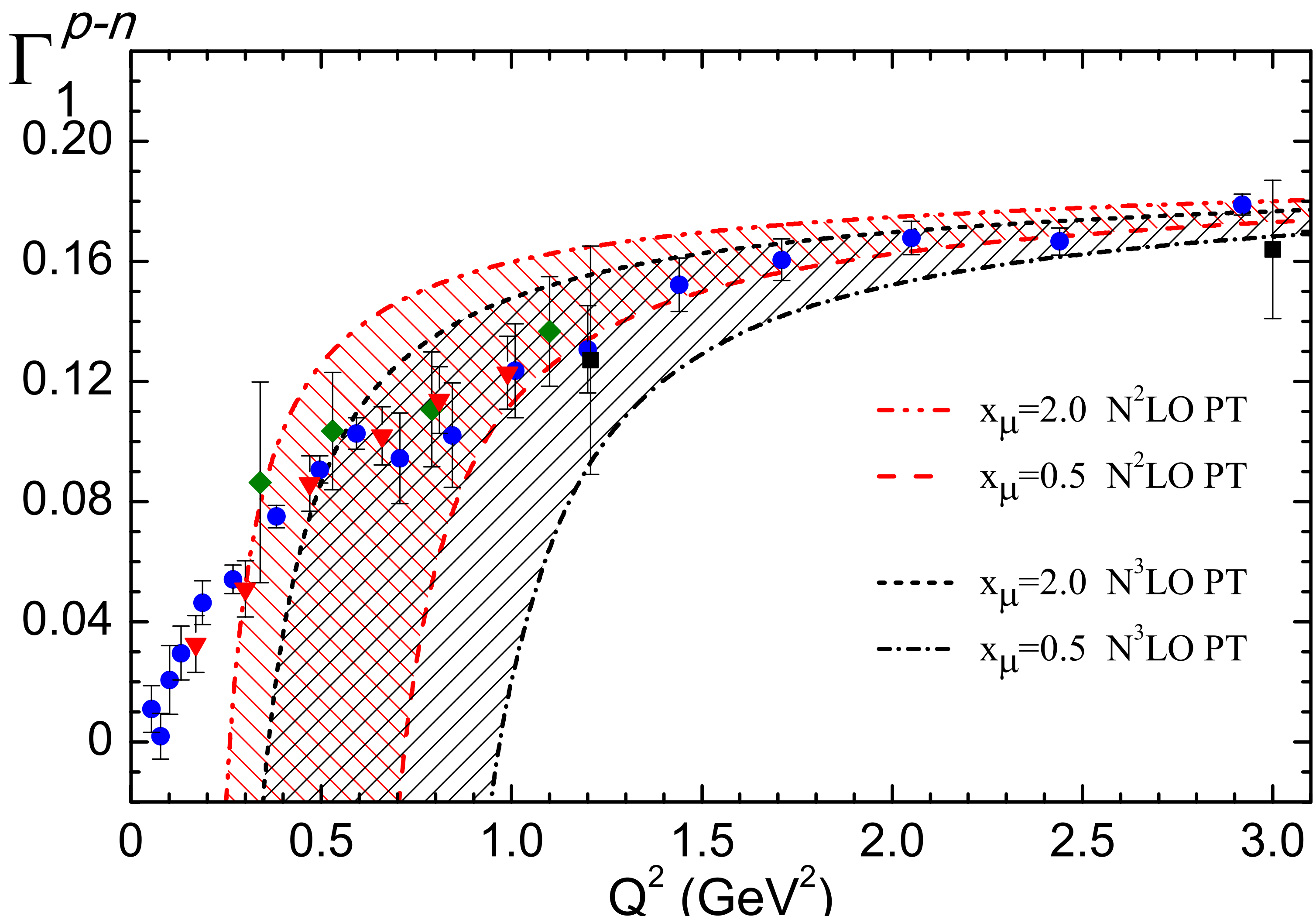
- Turbulence
- Classic and Quantum Statistics
- Quantum Fields

Reason – small parameter $g << 1$ at nonlinear structure

- Energy Gap in Semi-class and Quantum SuperFluidity
- Tunneling in QM (e.g., A.Vainstein,1966?)
- Quantum Fields, ...

Generally, a certain Asymptotic Series can correspond to various functions. Their "summation" is an Art.

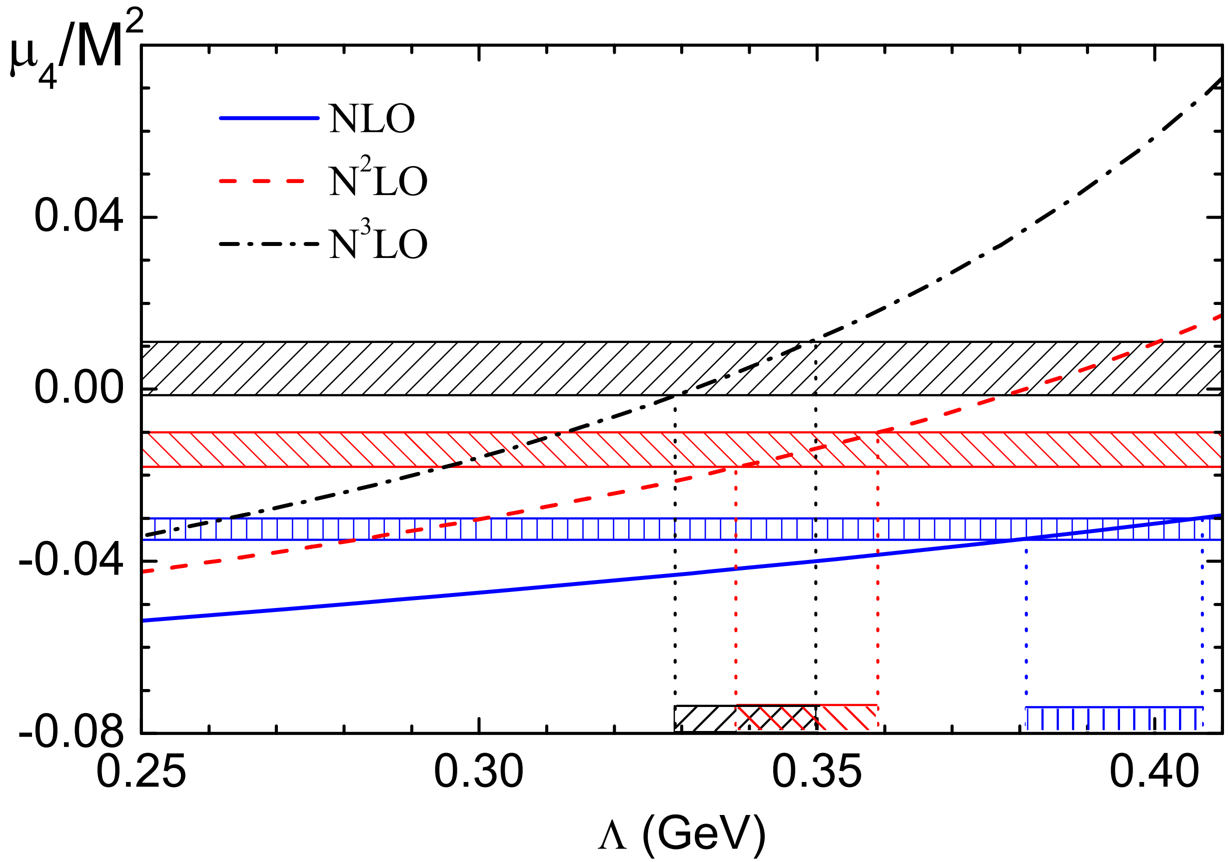
3- and 4-loop QCD for Bjorken Sum Rule



Description of JLab data for the 1st moment Γ_1^{p-n}

4-loop fit is slightly worse than the 3-loop

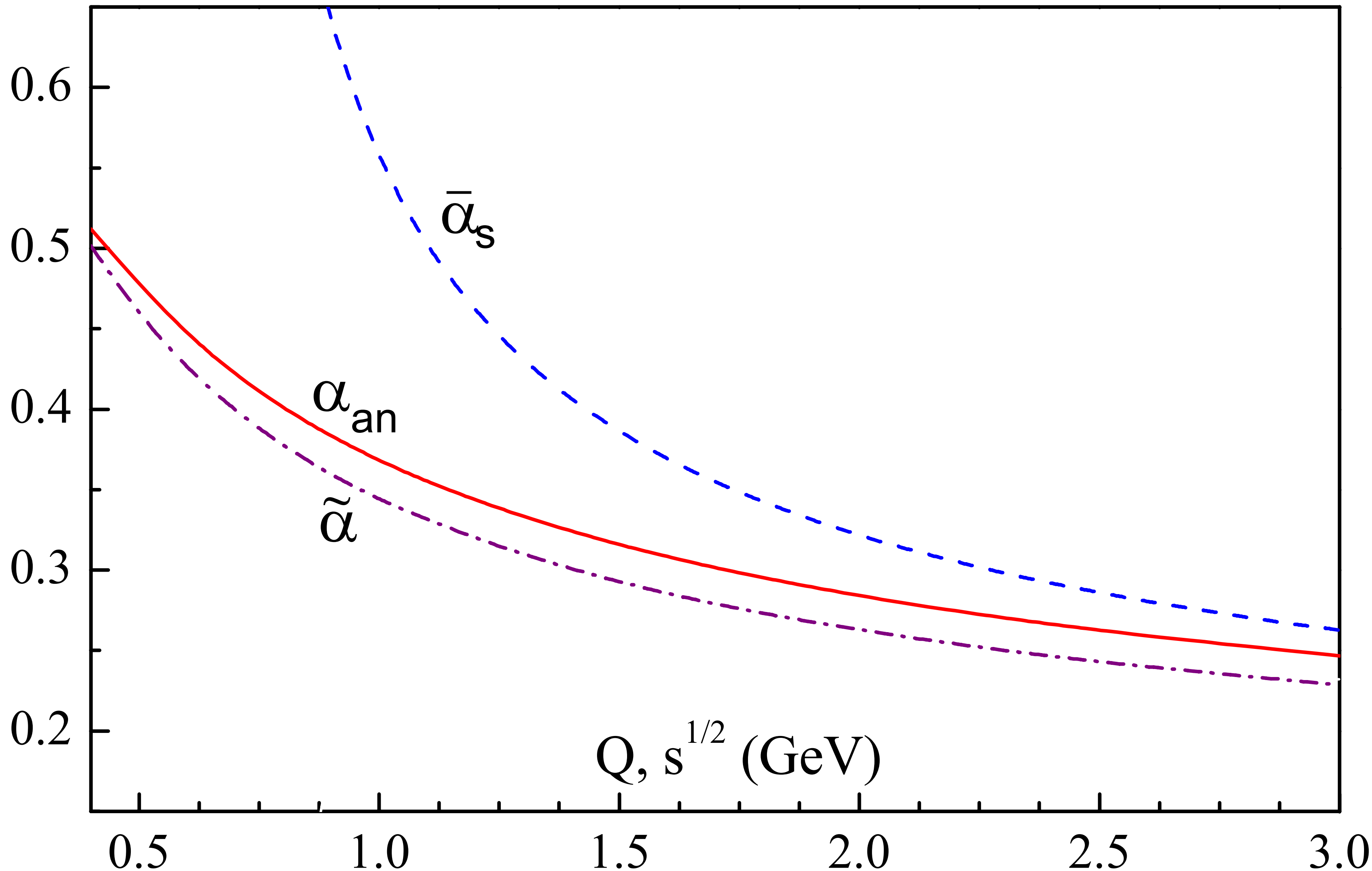
Extracting Λ_{QCD} from Bjorken Sum Rule



Extracting Λ_{QCD} from 3- and 4-loop fits for JLab data

Again no profit from the 4-loop fit !

Comparing APT couplings with $\alpha_s(Q^2)$



Few Words about APT

"Analytic Perturbation Theory"(APT) in QCD, the closed theor. scheme [Solovtsov+Sh-90s] without Landau-poles and additional parameters. It stems from imperative of Q^2-analyticity and compatibility with linear integral (like, Fourrier) transformations. Incorporates e^{-1/α_s} (algebraic in Q^2) structures.

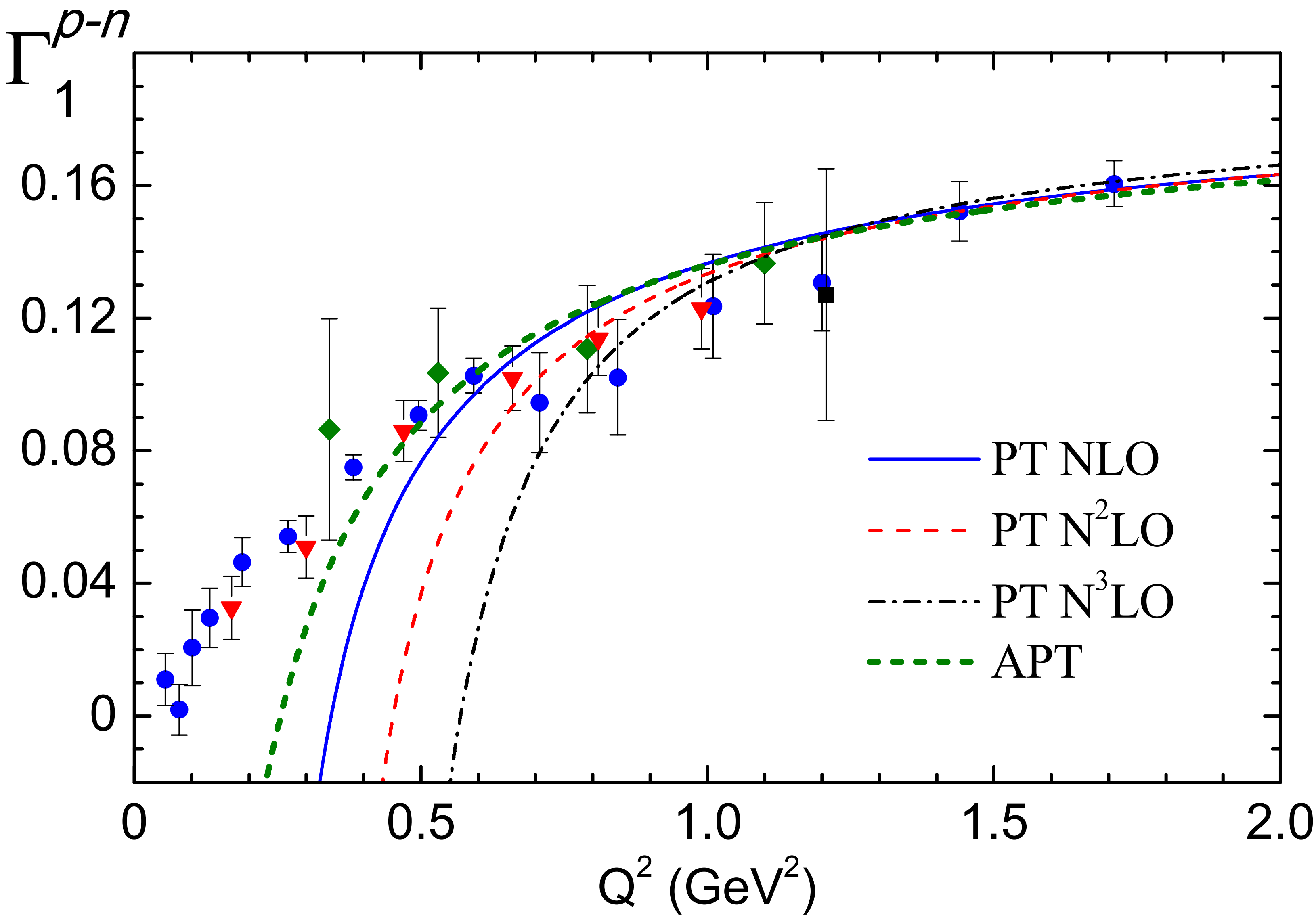
Instead of power PT set $\bar{\alpha}_s(Q^2), \bar{\alpha}_s(Q^2)^2, \bar{\alpha}_s(Q^2)^3, \ldots$ one has

non-power APT expansion set $\{\mathcal{A}_k(Q^2)\} k = 1, 2, \ldots$

with all $\mathcal{A}_k(Q^2)$ regular in the IR

$\mathcal{A}_1(Q)$ **quantitatively corresponds to Lattice Simulation results down to** $Q \sim 500\mathrm{MeV}$

The JLab-data Description by PT and by APT+HT



Anti-progress as 2– $\to 3- \to$ **4-loop below** $Q < 1\mathrm{GeV}$

On the $[Q^2\, exp 1/\alpha_s]$ structure

RG-invariance reduces the number of independent arguments

$$f(Q^2, \alpha_s) \to F_{RGinv}(\frac{1}{\alpha_s} + \beta_0 \ln\left(\frac{Q^2}{\mu^2}\right)) = \tilde{F}_{RG}\left(\frac{Q^2}{\mu^2}\, e^{1/\alpha_s}\right);$$

together with Q^2 analyticity yields one more statement on

inevitable not-perturb nature $\sim e^{-\dfrac{1}{\alpha_s}}$ of all algebraic -in Q^2- structures, like HT terms (and singularity-subtraction terms in APT)

Higher PT and APT contributions to observables

Relative contributions (in %) of 1– , 2–, 3– and 4–loop terms

Process		Scale/Gev	*PT(in %)*				*APT+HT* *		
Bjorken SR	t	1	35	20	19	**26**	80	19	1
Bjorken SR	t	1.78	56	21	13	**11**	80	19	1
GLS SumRule	t	1.78	65	24	**11**		75	21	4
Incl. τ-decay	s	1.78	51	27	14	7	88	11	1

* The 4-loop APT contributions are negligible everywhere

Higher PT term for $e^+e^- \longrightarrow$ hadrons

Relative contributions of 1- ... 4–loop terms in $e^+e^- \longrightarrow$ hadrons

Function	Scale/Gev	*PT terms (in %)*				Comment
r(s)	1	65	19	55	-39	**?!?**
r(s)	1.78	73	13	24	-10	**? ! ?**
d(Q)	1	56	17	11	16	
d(Q)	1.78	75	14	6	5	

Theoretical prediction of higher coefficients

Nice old example: The $g\,\phi^4$ beta-function was known up to the N^3LO term $\beta_{\overline{\rm MS}} = \frac{3}{2}\,g^2 - \frac{17}{6}g^3 + 16.27\,g^4 - 135.8\,g^5$

The Kazakov-Sh.-80 "summed" expression by Conform-Borel method

$$\beta^{CB}_{\overline{\rm MS}}(g) = \int_0^\infty \frac{dx}{g} e^{-x/g} \left(\frac{d}{dx}\right)^5 B(x) \quad \text{with} \qquad (1)$$

$$B(x) = a\,x^2(1 - b_2 w + \cdots - b_4 w^3); \quad w(x) - \text{conform variable}$$

contains N^4LO term $\beta^{CB}_6 = 1409.6\,.$

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$$\beta^{CB}_{\overline{\mathrm{MS}}}(g) = \int_0^{\infty} \frac{dx}{g} e^{-x/g} \left(\frac{d}{dx}\right)^5 B(x) \quad \text{with} \qquad (2)$$

$$B(x) = a\,x^2(1 - b_2 w + \cdots - b_4 w^3); \quad w(x) - \text{conform variable}$$

contains N^4LO term $\beta^{CB}_6 = 1409.6\,.$ Soon, it was calculated directly

$\beta_6 = 1420.6$ via Feynman diagrams. Comparing gives the amusing

accuracy of the CB (1) prediction – within 1 % !!

Outlook and Appeal

I. Invitation for Work

- Methods of summation, including integral and conformal tricks,
- Restoring Generating Function for HT terms
- Either generalizing minimal APT

II. Appeal for Speculating

- Toy models for the 4-loop term predicting for a process P_i
- Set of couplings $\alpha_s^i(Q^2)$ each adequate to the given process P_i ?
- Generating HT function for the each P_i ?

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- Repeating the Eidelman et al. (PL-1979) Borel approach ?