On relativization of the Sommerfeld-Gamow-Sakharov factor

A.B. Arbuzov[1,2], T.V. Kopylova[2]

[1] *BLTP, JINR, Dubna, Russia*

[2] *Dep. of higher mathematics, Dubna University, Dubna*

From Φ to Ψ, Novosibirsk, 19-22 September 2011

September 19, 2011

Motivation

- The problem of *non-perturbative* corrections and their matching to perturbative ones ...
- Relativistic two-particle problem ...
- Increasing experimental precision in observation of different e^+e^- annihilation channels
- Radiative return method allows scrutinizing the threshold region
- Intriguing experimental results on threshold enhancement in $e^+e^- \to \Lambda\bar{\Lambda}$
- SGS-like factor in QCD?
- Permanent interest of community and discussions in literature
- How to treat the factor within general-purpose computer codes?

The SGS factor features (I)

It's known for a long time that the re-scattering correction close to the threshold of a charged-particle-pair production is proportional to $|\Psi(0)|^2$ [A. Sommerfeld, *Atmobau und Spektralinien* (1921); J. Schwinger, *Particles, Sources, and Fields*, Vol.3.]

G. Gamow found the corresponding factor for the Coulomb barrier in nuclear interactions (1928)

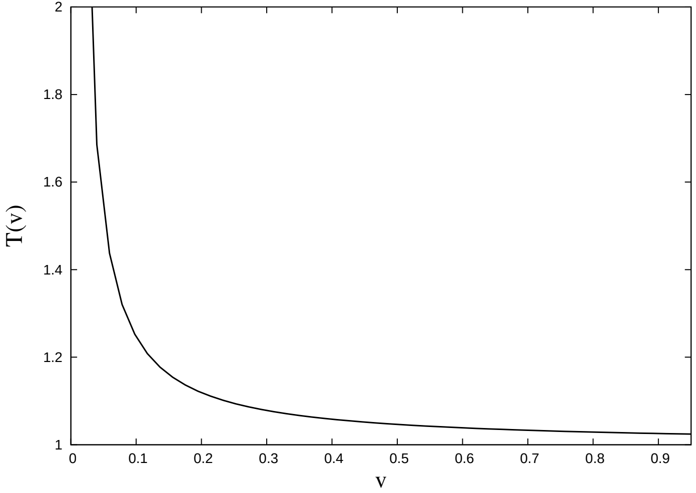
A. Sakharov considered just *Interaction of an electron and positron in pair production* (1948)

The Sommerfeld-Gamow-Sakharov (SGS) factor in the nonrelativistic approximation reads

$$T = \frac{\eta}{1 - e^{-\eta}}, \qquad \eta = -Q_1 Q_2 \frac{2\pi\alpha}{v}$$

where v is the relative velocity (*by construction*) of the particles with charges $Q_{1,2}$.

The SGS factor features (II)



The SGS factor features (III)

Perturbative expansion in α

$$T \approx 1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{3v^2} + \mathcal{O}\left(\frac{\alpha^3}{v^3}\right)$$

but for $v \to 0$ it breaks down

For opposite charges at very small v the factor behaves as

$$T\Big|_{Q_1 Q_2 = -1} \xrightarrow{v \to 0} \frac{2\pi\alpha}{v}$$

For equal charges at very small v the factor vanishes

$$T\Big|_{Q_1 Q_2 = 1} \xrightarrow{v \to 0} 0$$

Approaches to relativization of SGS factor (I)

- *Ad hoc*: $v \to 2\sqrt{1 - m^2/E^2} = 2\beta$

Approaches to relativization of SGS factor (I)

- *Ad hoc*: $v \to 2\sqrt{1 - m^2/E^2} = 2\beta$
- (Quasi)-relativistic quasi-potential equations

  It is possible to solve the two-particle problem with relativistic kinematics and interaction, but ...

Approaches to relativization of SGS factor (I)

- *Ad hoc*: $v \to 2\sqrt{1 - m^2/E^2} = 2\beta$
- (Quasi)-relativistic quasi-potential equations

  It is possible to solve the two-particle problem with relativistic kinematics and interaction, but ...
- Bethe-Salpeter equations (?)

Approaches to relativization of SGS factor (I)

- *Ad hoc*: $v \to 2\sqrt{1 - m^2/E^2} = 2\beta$
- (Quasi)-relativistic quasi-potential equations

  It is possible to solve the two-particle problem with relativistic kinematics and interaction, but ...
- Bethe-Salpeter equations (?)
- Resummation of (ladder) Feynman diagrams

Approaches to relativization of SGS factor (I)

- *Ad hoc*: $v \to 2\sqrt{1 - m^2/E^2} = 2\beta$
- (Quasi)-relativistic quasi-potential equations

  It is possible to solve the two-particle problem with relativistic kinematics and interaction, but ...
- Bethe-Salpeter equations (?)
- Resummation of (ladder) Feynman diagrams
- Extrapolation of (one-loop) perturbative calculations

Approaches to relativization of SGS factor (II)

Conditions on a relativized factor:

- Non-relativistic limit (must have)

Approaches to relativization of SGS factor (II)

Conditions on a relativized factor:

- Non-relativistic limit (must have)
- Ultra-relativistic limit $s \gg m_i^2 \;\Leftrightarrow\; v \to 1$ (desirable)

Approaches to relativization of SGS factor (II)

Conditions on a relativized factor:

- Non-relativistic limit (must have)
- Ultra-relativistic limit $s \gg m_i^2 \;\Leftrightarrow\; v \to 1$ (desirable)
- Heavy-mass limit $m_1 \ll m_2$ (very important)

Approaches to relativization of SGS factor (II)

Conditions on a relativized factor:

- Non-relativistic limit (must have)
- Ultra-relativistic limit $s \gg m_i^2 \;\Leftrightarrow\; v \to 1$ (desirable)
- Heavy-mass limit $m_1 \ll m_2$ (very important)
- Matching with perturbative calculations (?)

Quasi-potential relativistic equations (I)

There are many results in the literature on (quasi)relativistic two-particle eqs. Just look at $|\Psi(0)|^2$. Questions to these approaches always remain, but we can play there and understand the SGS factor better
In particular, in [A.A., Nuovo Cim.'1994] the case of scalar particles with arbitrary masses was evaluated

$$\frac{1}{i}\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\Psi = \left(\sqrt{\vec{p}_1^{\,2}+m_1^2}+\sqrt{\vec{p}_2^{\,2}+m_2^2}\right)\Psi$$

Use equal velocity reference frame: $\vec{p}_2 = -\vec{p}_1\ m_2/m_1 \ \Leftrightarrow\ \vec{v}_1 = -\vec{v}_2$.

Minimal substitution $p_i^\mu \to p_i^\mu - eA_i^\mu$ gives

$$\left[\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\rho + 1 - \frac{2\alpha}{v\rho} - \frac{l(l+1)}{\rho^2} + \frac{\alpha^2}{4\rho^2}(-1+\vec{u}^{\,2})\right] R_l(\rho) = 0$$

Quasi-potential relativistic equations (II)

For pure Coulomb interaction (A^0) we get

$$v_C = 2\sqrt{\frac{s - 4m^2}{s}} = 2\beta$$

Taking into account $\vec{A}$ leads to the relativistic relative velocity

$$v = \frac{\sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}}{s - m_1^2 - m_2^2}$$

The same result for $|\Psi(0)|^2$ was obtained also by I.T. Todorov [PRD'1971], H.W. Crater et al. [Ann.Phys.(NY)'1983, PRD'1992]

Obviously the limiting cases $m_1 \ll m_2$, $m_1 \gg m_2$ where we have exact solutions of the Klein-Gordon (and Dirac) equations are reproduced

Quasi-potential relativistic equations (III)

Jin-Hee Yoon and Cheuk-Yin Wong, *"Relativistic modification of the Gamow factor"* [PRC'2000, JPG'2005]:

$$T(v) \to K(v) = T(v) \cdot \kappa(v)$$

where κ depends on the type of particles *etc.*

O.P. Solovtsova and Yu.D. Chernichenko, *"Threshold resummation S-factor in QCD: the case of unequal masses"*, [Yad.Fiz.'2010]

Resummation of Feynman diagrams

V.N. Baier and V.S. Fadin, [ZhETF'1969] have shown that resummation of ladder-type Feynman diagrams for e^+e^- pair production leads to the factor in the form:

$$T_{\rm resum.} = \frac{\pi\alpha/\beta}{1 - e^{-\pi\alpha/\beta}} = T(2\beta)$$

Omitting diagrams with crossed photon lines corresponds to keeping only the Coulomb interactions. In this case we have an agreement with the quasi-potential picture

Resummation of non-ladder diagrams is difficult . . .

Perturbative calculations (I)

The SGS factor has a non-perturbative nature. But its expansion in α/v for $\alpha \ll v \ll 1$ makes sense

What goes on there in direct perturbative calculations?

Let's look at $\mathcal{O}(\alpha)$ FSR corrections to $e^+e^- \to \mu^+\mu^-$ with exact muon mass dependence, see *e.g.* A.A., D.Bardin, A.Leike [MPLA'1992], A.A. *et al.* [JHEP'2007]

The expansion in β starts from $\pi\alpha/\beta$ which has the correct non-relativistic limit, *i.e.* it agrees with the SGS factor expansion

What is the source of the Coulomb singularity in perturbative calculation?

What appears in the further expansion over β?

Perturbative calculations (II)

A. Hoang [PRD'1997]: one-loop contributions to moduli squared electric and magnetic form factors above the threshold:

$$\left(\frac{\alpha}{\pi}\right) g_e^{(1)}(s) \stackrel{\beta\to 0}{=} \frac{\alpha\pi}{2\beta} - 4\frac{\alpha}{\pi} + \frac{\alpha\pi\beta}{2} - \frac{4\alpha}{3\pi}\left[\ln\frac{m^2}{\lambda^2} + \frac{2}{3}\right]\beta^2 + \mathcal{O}\left(\beta^3\right)$$

$$\left(\frac{\alpha}{\pi}\right) g_m^{(1)}(s) \stackrel{\beta\to 0}{=} \frac{\alpha\pi}{2\beta} - 4\frac{\alpha}{\pi} + \frac{\alpha\pi\beta}{2} - \frac{\alpha}{3\pi}\left[4\ln\frac{m^2}{\lambda^2} - \frac{1}{3}\right]\beta^2 + \mathcal{O}\left(\beta^3\right)$$

The first and the third terms agree with the expansion of the SGS factor if $v = 2\beta/(1+\beta^2)$ *i.e.* the true relativistic relative velocity.

The second term comes from short distance ($\sim 1/m$) interactions. Factorization then gives

$$T(v)\cdot\left(1 - 4\frac{\alpha}{\pi}\right)$$

N.B. The picture is reproduced with the $\mathcal{O}\left(\alpha^2\right)$ form factors [R.Barbieri, J.A.Mignaco, E.Remiddi, Nuovo Cim.'1972]

Perturbative calculations (III)

After adding contributions of soft and hard photons the picture persists: nontrivial additional terms start to appear in $\mathcal{O}\left(\beta^3\right)$

The source of the singularity is the scalar triangular loop diagram:

$$\delta\sigma^{1-\mathrm{loop}} = \sigma^{\mathrm{Born}} \frac{\alpha}{2\pi}(s - m_1^2 - m_2^2) \cdot C_0(m_1^2, m_2^2, s, m_1^2, m_\gamma^2, m_2^2)$$

The pre-factor $(s - m_1^2 - m_2^2)$ and $C_0(\ldots)$ are universal for spinor, scalar and vector particles and for all partial waves involved in σ^{Born}

Direct calculations of C_0 for arbitrary masses gives an agreement to the expansion of the SGS factor with the proper relative relativistic velocity

$$v = \frac{\sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}}{s - m_1^2 - m_2^2}$$

Matching with perturbative calculations

We should **match** perturbative and non-perturbative results:

$$\sigma^{\mathrm{Corr.}} = \sigma^{\mathrm{Born}}\left(T(v) - \frac{\pi\alpha}{v} - \frac{\pi^2\alpha^2}{3v^2} - \dots \right) + \Delta\sigma^{1-loop} + \Delta\sigma^{2-loop} + \dots$$

For a cross check the matching should be always verified analytically by looking at the threshold behavior of the perturbative corrections

Unstable particle production

In the case of production of unstable particles, *e.g.* $t\bar{t}$ or W^+W^- one can not relay upon bound state effect ($|\Psi(0)|^2$) (multiple photon exchange) since the lifetime is comparable with the b.s. formation time, see *e.g.* V.S. Fadin and V.A. Khoze [Yad.Fiz.'1988], V.S. Fadin, V.A. Khoze and T. Sjostrand [Z.Phys.C'1990]

Nevertheless, direct perturbative calculations show the presence of the Coulomb singularity. See *e.g.* $z, \gamma^* \to W^+W^-$ in [D.Y.Bardin, W.Beenakker, A.Denner, PLB'1993].

N.B. Authors of this paper got

$$\beta = \frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{s}$$

But if we restore the factor being omitted there

$$\frac{s}{2(s-m_1^2-m_2^2)} \approx 1$$

we get exactly the relativistic relative velocity (one half)

Outlook

- Relativization of the SGS factor is not unique

Outlook

- Relativization of the SGS factor is not unique
- Using just the relativistic relative velocity in the standard formula looks preferable

Outlook

- Relativization of the SGS factor is not unique
- Using just the relativistic relative velocity in the standard formula looks preferable
- The factor is generalized for non-equal-mass case

Outlook

- Relativization of the SGS factor is not unique
- Using just the relativistic relative velocity in the standard formula looks preferable
- The factor is generalized for non-equal-mass case
- Numerically the effects for large v are subtle

Outlook

- Relativization of the SGS factor is not unique
- Using just the relativistic relative velocity in the standard formula looks preferable
- The factor is generalized for non-equal-mass case
- Numerically the effects for large v are subtle
- SGS factor is the same for different partial waves and spins

Outlook

- Relativization of the SGS factor is not unique
- Using just the relativistic relative velocity in the standard formula looks preferable
- The factor is generalized for non-equal-mass case
- Numerically the effects for large v are subtle
- SGS factor is the same for different partial waves and spins
- Matching with perturbative calculations should be taken into account